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SOME PROPERTIES OF

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ABSTRACT

In this paper we consider some properties of a broad class of network decomposition switches. Expression are given for the semi-Markov matrices of the output Markov renewal processes they produce, and a classification theorem for the states in these processes.

1. Introduction. We consider an arrival stream containing a number of different customer types, with J_n the type of the nth arrival. Customer C_n arrives at time T_n, with $0 = T_o < T_1 < \ldots$ We write $X_n = T_n - T_{n-1}$, with $X_o = 0$, and assume that J_n takes values in some countable set I. The switch is defined by the random variable Y_n, which takes one of a finite number, R, of values, depending on the output stream to which the nth customer is assigned.

In Cherry and McNickle [1] a switch $\{Y_n\}$ was defined to be semi-Markov if

$$P(J_n = j, Y_n = r, X_n \leq t \mid J_{n-1},\ldots J_o, Y_{n-1},\ldots Y_o, X_{n-1},\ldots X_1)$$

$$= P(J_n = j, Y_n = r, X_n \leq t \mid J_{n-1}, Y_{n-1}), \text{ for } n \geq 1.$$

The class of stationary semi-Markov switches was defined to be those semi-Markov switches with the additional property that

$$P(J_n = j, X_n \leq t \mid J_{n-1},\ldots J_o, Y_{n-1},\ldots Y_o, X_{n-1},\ldots X_1)$$

$$= P(J_n = j, X_n \leq t \mid J_{n-1}), \quad \text{for } n \geq 1.$$

2. Necessary Conditions for Semi-Markov Output Streams. In Cherry and McNickle [1] it was shown that the condition for a semi-Markov switch was sufficient to ensure that each of the output streams formed a Markov renewal process. We may ask if such a condition is necessary.

Although all of the published examples of switches are semi-Markov it turns out that the condition is not necessary, as the following example shows.

Consider a renewal process with distribution function F(t), which forms the input to a switch producing three output streams. Customers are assigned deterministically in the order ...1,2,3,2,1,2,3. ... Now

$$P(Y_n = 3, X_n < t \mid Y_{n-1} = 2, Y_{n-2}, \ldots Y_o) = \begin{cases} 0, & Y_{n-2} = 3, \\ F(t), & Y_{n-2} = 1, \end{cases}$$

and so the switch is not semi-Markov, yet clearly streams 1 and 3 are renewal processes with common distribution function $F^4(t)$, and stream 2 is a renewal process with distribution function $F^2(t)$. Here $F^n(t)$ stands for the n-fold convolution of $F(t)$. We have the following result, however, at least when the input process has non-lattice distributions.

THEOREM 1 Any stationary switch which acts on a Markov renewal process to produce output streams which are Markov renewal processes defined on the same state space has a representation as a semi-Markov switch.

PROOF. Consider the sequence of events:

$$J_{n-R}, Y_{n-R}, X_{n-R}, \ldots\ldots\ldots J_n, Y_n = r, X_n; \text{ for some } r \in \{1,2,\ldots,R\}$$

Either: (i) At least one of the arrivals $C_{n-1}, \ldots, C_{n-R}$, (say C_ℓ), was also assigned to the r-th output stream, so that with probability one

$$P(J_n = j, Y_n = r, X_n \le t \mid J_{n-1}, Y_{n-1}, X_{n-1}, \ldots, Y_\ell = r, X_{\ell-1}, \ldots X_{n-R}, \ldots, Y_o)$$

$$= P(J_n = j, Y_n = r, X_n \le t \mid J_{n-1}, Y_{n-1}, X_{n-1}, \ldots, Y_\ell = r),$$

since the r-th output stream is a renewal process;

Or: (ii) At least two of the arrivals $C_{n-1}, \ldots, C_{n-R}$ (say C_p and C_q), were assigned to the same output stream, and hence for some $s \in \{1,2,\ldots,R\}$:

$$P(J_n = j, Y_n = r, X_n \le t \mid J_{n-1}, Y_{n-1}, \ldots, Y_q = s, \ldots, Y_p = s, \ldots, X_{n-R}, \ldots, Y_o)$$

$$= P(J_n = j, Y_n = r, X_n \le t \mid J_{n-1}, Y_{n-1}, X_{n-1}, \ldots, Y_p = s).$$

Now $P(J_n = j, Y_n = r, X_n \le t \mid J_{n-1}, Y_{n-1} = s, X_{n-1}, \ldots, J_{n-R}, Y_{n-R})$

$$= P(J_n = j \mid J_{n-1}, Y_{n-1} = s, X_{n-1}, \ldots, J_{n-R}, Y_{n-R}) \cdot$$

$$P(Y_n = r, X_n \le t \mid J_n = j, J_{n-1}, Y_{n-1} = s, X_{n-1}, \ldots, J_{n-R}, Y_{n-R}),$$

and since both $\{J_n, X_n\}$ and the s th output stream are Markov renewal processes, the event $\{J_n = j, Y_n = r, X_n \leq t\}$ must be conditionally independent of $X_{n-1}, X_{n-2}, \ldots$

So since by total probability the switch is independent of (J_{n-R}, Y_{n-R}), the switch must be equivalent to one in which $\{J_n, Y_n, \ldots J_{n-R+1}, Y_{n-R+1}, X_n\}$ forms a Markov renewal process. Hence it has a representation as a semi-Markov switch.

Thus for example the three state switch mentioned previously can be considered a semi-Markov switch with an imbedded Markov chain of order two.

A useful case of the theorem is when $R = 2$, giving the following corollary.

COROLLARY: A stationary switch which acts on a Markov renewal process to produce two output streams which are Markov renewal processes defined on the same state space must be semi-Markov.

3. The Output Streams

We assume for convenience that $\{J_n, Y_n, X_n\}$ is an irreducible Markov renewal process and that the initial arrival was assigned to the r-th output stream. Events in the filter set $K = I \times \{r\}$ of the state space of the Markov renewal process forms the r-th output stream.

In particular, let

$$n_o = 0,$$

$$n_{k+1} = \inf(i > n_k : (J_i, Y_i) \in K), \; k = 0,1,2,\ldots,$$

and define the process $\{Z_k, \tau_k\}$ by:

$$Z_k = J_{n_k}, \quad \tau_k = T_{n_k}, \quad \text{for } k = 0,1,2,\ldots$$

We have shown previously that $\{Z_k, \tau_k\}$ is a Markov renewal process which describes the behaviour of the r-th output stream.

Let $G(t) = \{G_{ij}(t)\} = \{P(Z_{k+1} = j, W_{k+1} \leq t \mid Z_k = j)\}$, $(i,j) = 1, \ldots m$.

where $W_{k+1} = \tau_{k+1} - \tau_k$. A formal expression for the semi-Markov matrix of the r-th output stream, $G(t)$, may be found from Cinlar [3] .

We write the semi-Markov matrix of the $\{J_n, Y_n, X_n\}$ process in block form as $A = \{A_{pq}\}$, where the i,j-th element of the m x m matrix $A_{pq}(t)$ is $P(J_n = j,\ Y_n = q,\ X_n \leq t \mid J_{n-1} = i,\ Y_{n-1} = p)$. Then on relabelling the output streams if necessary, A may be partitioned as

$$A = \begin{bmatrix} A_{rr} & B \\ C & D \end{bmatrix}.$$

Cinlar's analysis then shows that $G(t)$ then satisfies:

$$G(t) = A_{rr}(t) + B * (\sum_n D^n) * C(t),$$

where $*$ stands for the usual matrix convolution operation and D^n for the n-fold convolution of D with itself.

If the first arrival was assigned to the s-th output stream, where $s \neq r$, then partitioning A as:

$$A = \begin{bmatrix} A_{sr} & B' \\ C' & D' \end{bmatrix},$$

leads to a similar expression for the matrix of transition functions for the time until the first event in the r-th output stream.

It appears, however, that no simplification of these expressions can be made unless the class of switches is restricted further.

Switches depending only on the input Markov renewal process.

Consider the class of stationary semi-Markov switches in which the switch no longer depends explicitly on the assignment of the last customer. Thus $P(Y_n = r \mid J_n, X_n, J_{n-1}, Y_{n-1}) = P(Y_n = r \mid J_n, X_n, J_{n-1})$, for n = 1,2,...

The semi-Markov matrices for the output streams of a special case of this class where arrivals are assigned according to type alone were derived directly in Cinlar [2]. By considering a suitable Markov renewal equation expressions for those of all switches of this class can be found.

Since the switch is stationary, for any $s \in \{1,2,\dots,R\}$

$$P(J_n = j,\ Y_n = r,\ X_n \leq t \mid J_{n-1} = i,\ Y_{n-1} = s) = f_{ijr}(t)$$

$$= f_{ij}(t)\, q_r(i,j,t),$$

where $F(t) = \{f_{ij}(t)\}$ is the semi-Markov matrix of the arrival process, and

$$q_r(i,j,t) = P(Y_n = r \mid J_n = j,\ X_n \leq t,\ J_{n-1} = i).$$

Let $G(t) = \{G_{ij}(t)\}$ be the semi-Markov matrix for the r-th output stream. Then since the condition on the switch implies that all the higher order transition probabilities of the $\{J_n, Y_n, X_n\}$ process are independent of the initial state of the switch

$$G_{ij}(t)$$

$$= \sum_{\ell=1}^{\infty} P\,(J_{n_{k+1}} = j,\ n_{k+1} - n_k = \ell,\ T_{n_{k+1}} - T_{n_k} \leq t \mid J_{n_k} = i)$$

$$= f_{ijr}(t) + \sum_{\ell=2}^{\infty} \sum_{h=1}^{m} \sum_{\substack{s=1 \\ s \neq r}}^{R} \int_0^t P\,(J_{n_k+1} = h, Y_{n_k+1} = s, X_{n_k+1} \in (x, x+dx) \mid J_{n_k} = i).$$

$$P\,(J_{n_{k+1}} = j,\ n_{k+1} - n_k - 1 = \ell - 1, T_{n_{k+1}} - T_{n_k+1} \leq (t-x) \mid J_{n_k+1} = h, Y_{n_k+1} = s)$$

$$= f_{ijr}(t) + \sum_{\ell=2}^{\infty} \sum_{h=1}^{m} \int_0^t (f_{ih}(dx) - f_{ihr}(dx)).$$

$$P\,(J_{n_{k+1}} = j,\ n_{k+1} - n_k - 1 = \ell - 1,\ T_{n_{k+1}} - T_{n_k+1} \leq (t-x) \mid J_{n_k+1} = h).$$

Hence

$$G_{ij}(t) = f_{ijr}(t) + \sum_{h=1}^{m} \int_0^t (f_{ih}(dx) - f_{ihr}(dx))\, G_{hj}(t-x),$$

for all $i,j \in I$. (1)

If the first arrival to the switch was not assigned to the r-th output stream then it can be seen that the conditions on this class of switch mean that the distributions of the delay until the first event in the r-th output stream also satisfy the equations (1). These equations can be solved under a further weak assumption to give:

Theorem 3

If all the states of the input Markov renewal process are conservative then the r-th output stream from the switch forms a Markov renewal process with semi-Markov matrix and matrix-valued distribution function of the time until the first event both given by

$$G(t) = \int_0^t R_r(dx)\, F_r(t-x) \ , \qquad (2)$$

where $F_r(t) = \{f_{ijr}(t)\}$, and $R_r(t)$ is the Markov renewal matrix corresponding to the semi-Markov matrix $\{f_{ij}(t) - f_{ijr}(t)\}$.

Proof: $F_r(t)$ and $\{f_{ij}(t) - f_{ijr}(t)\}$ are both semi-Markov matrices, so (1) is a Markov renewal equation. If all states in I are conservative then Cinlar [3] shows that a unique solution to equations of this type exists, and in this case is given by (2).

If I is finite, then at least for $Re(s) > 0$

$$\underline{G}(s) = \sum_{n=0}^{\infty} (\underline{F}(s) - \underline{F}_r(s))^n\, \underline{F}_r(s),$$

$$= \quad (I - \underline{F}(s) + \underline{F}_r(s))^{-1}\ \underline{F}_r(s), \tag{3}$$

where $\underline{F}_r(s) = \int_0^\infty e^{-st} F_r(dt)$, $\quad \underline{G}(s) = \int_0^\infty e^{-st} G(dt)$.

We can further note that if $f_{ijr}(t) = f_{ij}(t).q_r(j)$, so that the switch depends on the type of the arrival only, then with $Q = \{\delta_{ij} q_r(j)\}$

$$\begin{aligned} \underline{G}(s) &= (I - \underline{F}(s) + \underline{F}(s).Q)^{-1}\underline{F}(s).Q, \\ &= \underline{F}(s)(I - (I-Q)\underline{F}(s))^{-1}.Q \ , \end{aligned}$$

which was Cinlar's 1969 result.

One example of the application of (3) is to a generalization of Palm's overflow problem to the SM/M/1/1 queue. Customers arrive according to a Markov renewal process at a negative exponential server with mean service time $\frac{1}{\mu}$. If there is no waiting room then the stream of customers who join the queue and those who overflow can be considered as the results of a semi-Markov switch of this type. For the overflow stream $F_r(s) = F(s+\mu)$, and so

$$G(s) = (I - F(s) + F(s+\mu))^{-1} F(s+\mu)$$

is the semi-Markov matrix of the overflow stream. The renewal case of this result is well known e.g. see Khinchine [4] .

4. <u>Classification of states in the output streams</u>. Although states which are transient in the arrival process will clearly also be transient in any output stream, the same is not true for recurrent states.

Let $f_{ijrs} = P(J_n = j,\ Y_n = s \mid J_{n-1} = i,\ Y_{n-1} = r)$, with $G = \{g_{ij}\}$ the imbedded Markov chain for the rth output stream. We write (j,r) for the state $(J_n = j,\ Y_n = r)$.

The following result completes the characterization of the output streams of a semi-Markov switch.

THEOREM 4 For a stationary semi-Markov switch state j in the r-th output stream will be recurrent if and only if there exists some recurrent state k in the arrival process such that $j \to k$ and if $(j,r) \dashrightarrow (k,s)$ for any s then $(k,s) \longrightarrow (j,r)$.

PROOF Clearly $\sum_{p=1}^{\infty} g_{jj}^{p} = \sum_{\ell=1}^{\infty} f_{jjrr}^{\ell}$.

Let $R' = \{s \mid (j,r) \dashrightarrow (k,s)\}$, with $N(s) = \min\,(n \mid f_{kjsr}^{n} > 0)$, $N = \max_{R'}\,(N(s))$, and $\delta = \min_{R'}\,(f_{kjsr}^{N(s)})$.

$$\begin{aligned}
\sum_{p=1}^{\infty} g_{jj}^{p} &\geq \sum_{\ell=N}^{\infty} \sum_{k=1}^{m} \sum_{s=1}^{R} f_{jkrs}^{\ell-N(s)}\, f_{kjsr}^{N(s)}, \\
&\geq \sum_{\ell=N}^{\infty} \sum_{R'} f_{jkrs}^{\ell-N(s)}\, f_{kjrs}^{N(s)}, \\
&\geq \delta \sum_{\ell=N}^{\infty} \sum_{R'} f_{jkrs}^{\ell-N(s)}, \\
&= \delta \sum_{\ell=N}^{\infty} \sum_{s=1}^{R} f_{jkrs}^{\ell-N(s)},
\end{aligned}$$

which is divergent, since k is recurrent and $j \longrightarrow k$.

Necessity of the first condition follows since if (j,r) is recurrent then at least one of the terms $f_{jkrs}^{\ell-N(s)}\, f_{kjsr}^{N(s)}$ must be positive, and hence the recurrent state k exists. The second condition is a standard result for recurrent states.

If the switch is of the type described in section 4 then since $\sum_{p=1}^{\infty} g_{ij}^{p} = \sum_{\ell=1}^{\infty} \sum_{k=1}^{m} f_{jk}^{\ell-1}\, f_{kjr}$, where $f_{kjr} = f_{kjr}(\infty)$, the conditions of the theorem may be simplified.

COROLLARY: For a semi-Markov switch in which the assignment of an arrival is independent of the previous assignment, state j in the rth output stream is recurrent if and only if there exists some recurrent state k in the arrival process such that $k \longrightarrow j$ and $f_{kjr} > 0$.

The proof of this is similar to that of the theorem and is ommitted.

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